Simple Harmonic Motion
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Question

This question is concerned with the oscillations of a single particle, mass m, moving on a smooth horizontal table under the influence of two or more perfect springs. Each spring obeys Hooke's law with stiffness k and has natural length l_0. In all diagrams the system is in its equilibrium position.

For the system comprising just one spring with motion in the direction x, as shown in figure 1,

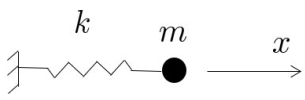

Figure 1: Single spring-mass system.

the period, τ, is, in terms of k and m, given by

$$\tau = 2\pi\sqrt{\frac{m}{k}}.$$

In the following parts you are asked to obtain the appropriate period, τ_i, in the form

$$\tau_i = \lambda_i \tau.$$

In each case the equilibrium position is defined by the length L with $L > l_0$.

(i) Consider the system with two springs as shown in figure 2.

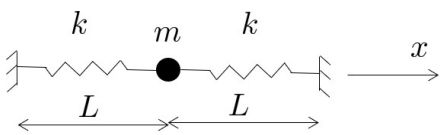

Figure 2: System with two springs along direction of motion.

Find the value of λ_1.

[2 marks]

For the remaining parts you should consider *small amplitude* oscillations in the appropriate direction x, i.e. $|x|/L << 1$. You may find it convenient to write $X = x/L$ and evaluate distances correct to first order in X, i.e. ignore terms X^2 and higher, and then write down the appropriate equation of motion.

(ii) This part involves two springs as shown in figure 3.

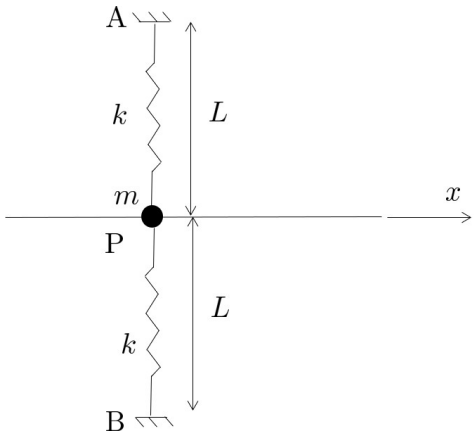

Figure 3: System with two springs perpendicular to direction of motion.

Find the value of λ_2.

[3 marks]

(iii) You may find the result in this section to be of help in sections (iv) and (v).

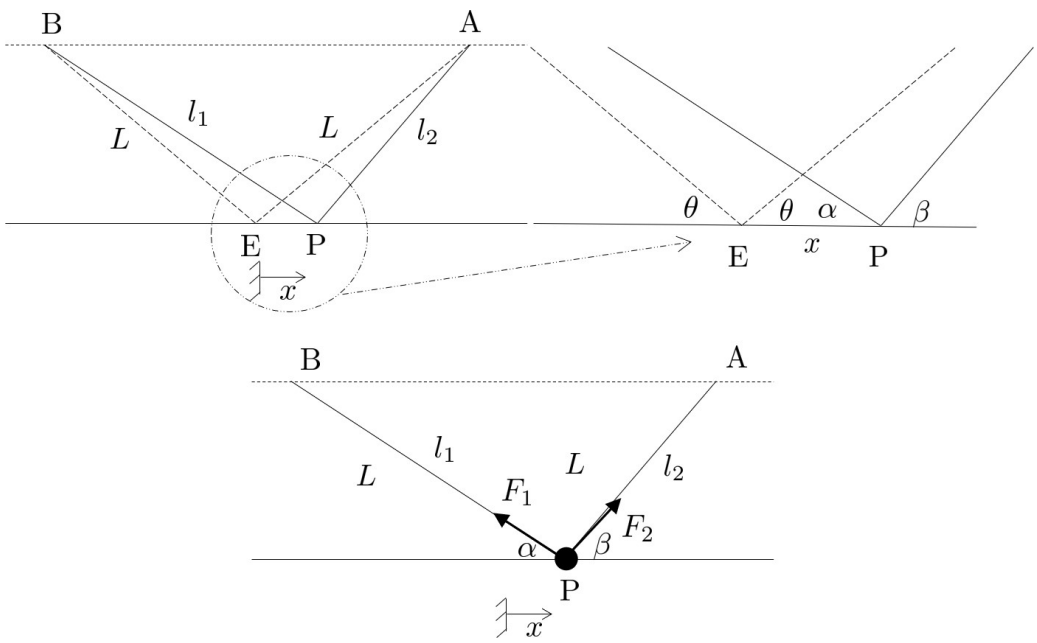

Figure 4: (a) Definition of l_1, l_2, α and β; (b) forces F_1 and F_2.

The situation shown in figure 4(a) shows a point, P, a *small* distance, x, from the point E. Show:

Either

$$l_1 = L\,(1 + X\cos\theta) + \mathcal{O}(X^2) \text{ and } \cos\alpha = \cos\theta + X\left(1 - \cos^2\theta\right) + \mathcal{O}(X^2).$$

Or

$$l_2 = L\,(1 - X\cos\theta) + \mathcal{O}(X^2) \text{ and } \cos\beta = \cos\theta + X\left(\cos^2\theta - 1\right) + \mathcal{O}(X^2).$$

Suppose that we have a symmetric system of springs such that the motion is always along the direction x as shown in figure 4. Two such springs are shown in figures 4(a) and (b).

Using the results above, show that the magnitude of the force in each spring is given, to first order, by

$$F_1 = F_2 = kx\cos\theta$$

and that the tension force, F_1, and the compression force, F_2, yield a combined, first order, component in the x-direction

$$-2kx\cos^2\theta.$$

[2 marks]

(iv) This part involves four springs along the diagonals of a square, ABCD, as shown in figure 5.

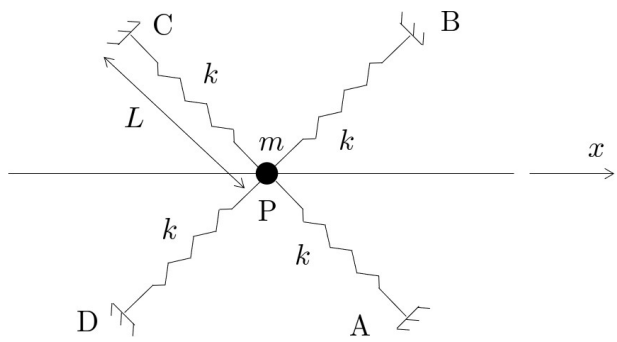

Figure 5: System with four springs along the diagonals of a square.

Find the value of λ_3.

[2 marks]

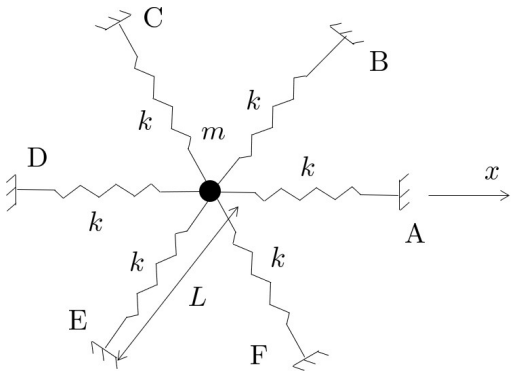

Figure 6: System with six springs in a regular hexagon.

(v) This final part involves six springs arranged in a regular hexagonal form, ABCDEF, as shown in figure 6.

Find the value of λ_4.

[1 mark]

[Total 10 marks]

Solution
Either:

(A) *Using Newton's second law*

(i) (2 marks) In figure 7 the displacement, x, is measured relative to the equilibrium position x_e. The forces, F_1 and F_2, are the forces *above and beyond* the equilibrium values[1]. It is these forces which give rise to the periodic motion.

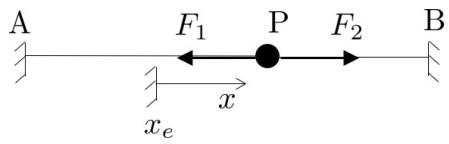

Figure 7: Particle P relative to the equilibrium position x_e.

Newton's second law gives

$$m\ddot{x} = -F_1 + F_2.$$

Now in the configuration shown in figure 7, F_1 is a tension and F_2 is a compression. Hence Hooke's law gives

$$F_1 = kx \text{ and } F_2 = -kx.$$

It follows that the equation of motion is

$$m\ddot{x} = -2kx$$

or

$$\ddot{x} = -{\omega_1}^2 x, \text{ where } {\omega_1}^2 = \frac{2k}{m},$$

giving

$$\tau_1 = 2\pi\sqrt{\frac{m}{2k}}$$

and consequently

$$\lambda_1 = \frac{1}{\sqrt{2}}.$$

[2 marks]

[1]Suppose that the forces are written ${F_1}^{total} = {F_1}^{eq} + F_1$ and ${F_2}^{total} = {F_2}^{eq} + F_2$.
Newtons' second law gives $-{F_1}^{total} + {F_2}^{total} = m\ddot{x}$.
Since $-{F_1}^{eq} + {F_2}^{eq} = 0$ it follows that $m\ddot{x} = -F_1 + F_2$.

(ii) (3 marks) With $X = x/L$ consider the two spring set up displaced a small distance x as shown in figure 8. Here $|X| << 1$ and we see that

$$\text{AP} = \text{BP} = L\sqrt{1 + X^2}.$$

It follows then that

$$\text{AP} = \text{BP} = L\left(1 + \frac{X^2}{2} + \mathcal{O}(X^4)\right)$$

and

$$F_1 = F_2 \approx k\left[L\left(1 + \frac{X^2}{2}\right) - l_0\right].$$

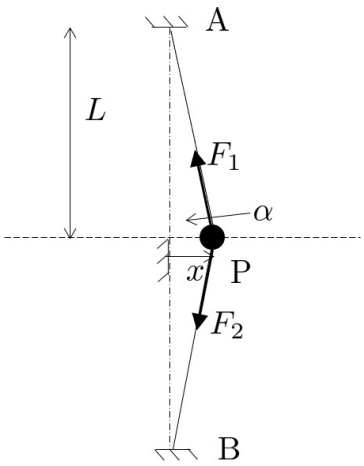

Figure 8: System with two springs perpendicular to direction of motion.

It follows that the tensions in each spring are, to first order,

$$F_1 = F_2 = k\left(L - l_0\right),$$

the equilibrium tension.

From figure 8 we see that $\cos\alpha = X/\sqrt{1 + X^2} = X$, to first order.

Newton's second law gives

$$m\ddot{x} = -F_1\cos\alpha - F_2\cos\alpha$$

which we can write, to first order, as

$$m\ddot{X} = -2k\left(1 - \frac{l_0}{L}\right)X$$

i.e.

$$\ddot{X} = -{\omega_2}^2 X, \text{ where } {\omega_2}^2 = \frac{2k}{m}\left(1 - \frac{l_0}{L}\right),$$

giving

$$\tau_2 = 2\pi\sqrt{\frac{m}{2k\left(1 - \frac{l_0}{L}\right)}}$$

and consequently

$$\lambda_2 = \sqrt{\frac{1}{2\left(1 - \frac{l_0}{L}\right)}}.$$

[3 marks]

(iii) *Either*
(2 marks) Using the cosine rule on triangle EPB, figure 4

$$\begin{aligned} l_1 &= \sqrt{L^2 + x^2 + 2Lx\cos\theta} \\ &= L\left(1 + 2X\cos\theta + X^2\right)^{1/2} \\ &= L\left(1 + X\cos\theta\right) + \mathcal{O}(X^2). \end{aligned}$$

Also, we can see that

$$\begin{aligned} \cos\alpha &= \frac{L\cos\theta + x}{l_1} \\ &= \left(\cos\theta + X\right)\left(1 + X\cos\theta\right)^{-1} \\ &= \cos\theta + X\left(1 - \cos^2\theta\right) + \mathcal{O}(X^2). \end{aligned}$$

Or
Similarly, using the cosine rule on triangle EPA, figure 4

$$\begin{aligned} l_2 &= \sqrt{L^2 + x^2 - 2Lx\cos\theta} \\ &= L\left(1 - 2X\cos\theta + X^2\right)^{1/2} \\ &= L\left(1 - X\cos\theta\right) + \mathcal{O}(X^2). \end{aligned}$$

Also, we can see that

$$\begin{aligned} \cos\beta &= \frac{L\cos\theta - x}{l_2} \\ &= \left(\cos\theta - X\right)\left(1 - X\cos\theta\right)^{-1} \\ &= \cos\theta + X\left(\cos^2\theta - 1\right) + \mathcal{O}(X^2). \end{aligned}$$

[2 marks]

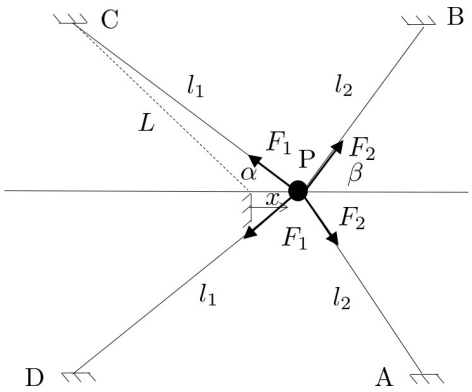

Figure 9: System with four springs.

(iv) (2 marks) By symmetry, CP = DP = l_1 and the forces in the springs spring are equal (tensions F_1). Similarly for BP and AP (length l_2 and compressions F_2), see figure 9. With $X = x/L$ consider the four spring set up displaced a small distance x as shown in the figure. Here $|X| << 1$.

Using the notation of figure 4

$$\theta = \pi/4$$

and, using the results of part (iii), the total force over and above equilibrium value is given by

$$2\left(-2kx\cos^2\left(\frac{\pi}{4}\right)\right) = -2kx.$$

Newton's second law then gives

$$m\ddot{x} = -2kx$$

so that

$$\tau_3 = 2\pi\sqrt{\frac{m}{2k}}$$

and

$$\lambda_3 = \frac{1}{\sqrt{2}}.$$

(v) (1 mark) In this case we have, in terms of notation of figure 4, two springs with $\theta = 0$, and four with $\theta = \pi/3$ so that using the result of part (iii) the total force over and above equilibrium value is given by

$$-2kx + 2\left(-2kx\cos^2\left(\frac{\pi}{3}\right)\right) = -3kx$$

Newton's second law then gives

$$m\ddot{x} = -3kx$$

so that

$$\tau_4 = 2\pi\sqrt{\frac{m}{3k}}$$

and

$$\lambda_4 = \frac{1}{\sqrt{3}}.$$

Or:

(B) *Using conservation of energy*

(i) From figures 2 and 7 we can obtain the potential energy as

$$\frac{1}{2}k\left(L + x - l_0\right)^2 + \frac{1}{2}k\left(L - x - l_0\right)^2$$

and conservation of energy yields

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}k\left(L + x - l_0\right)^2 + \frac{1}{2}k\left(L - x - l_0\right)^2 = \text{constant}.$$

Differentiate with respect to t

$$m\ddot{x}\dot{x} + k\left(L + x - l_0\right)\dot{x} + k\left(L - x - l_0\right)\dot{x} = 0.$$

Dividing by $\dot{x}$ gives

$$m\ddot{x} = -2kx,$$

leading to

$$\lambda_1 = \frac{1}{\sqrt{2}}$$

as before.

[2 marks]

(ii) From figures 3 and 8 we can obtain the potential energy as

$$2.\tfrac{1}{2}k\left(\sqrt{L^2 + x^2} - l_0\right)^2$$

and conservation of energy yields

$$\frac{1}{2}m\dot{x}^2 + k\left(L^2 + x^2 + l_0^2 - 2l_0\left(L^2 + x^2\right)^{\frac{1}{2}}\right) = \text{constant}.$$

Differentiate with respect to t

$$m\ddot{x}\dot{x} + 2kx\dot{x} - 2kl_0\tfrac{1}{2}\frac{2x\dot{x}}{\left(L^2 + x^2\right)^{\frac{1}{2}}} = 0$$

which we acn write as, after division by $\dot{x}$

$$m\ddot{x} + 2kx - 2k\frac{l_0}{L}x\left(1 + \frac{1}{2}X^2 + \mathcal{O}(X^4)\right) = 0$$

and so, to first order

$$m\ddot{x} = -2k\left(1 - \frac{l_0}{L}\right)x$$

giving

$$\lambda_2 = \sqrt{\frac{1}{2\left(1 - \frac{l_0}{L}\right)}}$$

as before.

[3 marks]

(iii) This part requires a purely geometric approach and so conservation of energy is not appropriate.

(iv) and (v)
These two parts can be done using conservation of energy, however it requires a very large amount of tedious algebra and it is not shown here.