Solutions

Hemisphere hydrodynamics

For part one (4 marks), it is needed to get the expression of the hydrostatic force acting on the funnel. At the stage when it gets larger than the funnel gravitational force, the funnel takes off the surface and the liquid starts leaking.

Let us consider a force acting at an infinitesimal spherical band at the height z. Refer to the figure below for notations and angles. From geometrical consideration, the angle ϕ between the force $d\vec{N}$ and horizontal is $\sin\phi = z/R$. The force magnitude is a product of hydrostatic pressure and band's area: $dN = p(z)\,dS$. The pressure at the depth is given as $\rho g h$. If the origin is located at the centre of the funnel, and the total depth is designated as H, pressure at coordinate z is given as $p((z) = \rho g(H - z)$

For detalisation, we can derive the exact expression for the band's surface area. However, the force's vertical component is proportional to the horizontal projection's surface area on the band (its effective shadow while illuminated from the top). It simplifies the derivation to some extent. In the general case, the spherical band's area is given as $S = 2\pi r h$. In this application, it transforms to $dS = 2\pi R dz$. The expression for the force is therefore:

$$dN = p(z)\,dS = \rho g(H - z) \cdot 2\pi R\,dz.$$

Its vertical componets is given as (refer to the plotting):

$$dN_z = dN \cdot \sin\phi = \rho g(H - z) \cdot 2\pi \cancel{R} \frac{z}{\cancel{R}}\,dz.$$

To find the total force value, we intergate over the total depth of the liquid $z = 0$ to $z = H$. On the other hand, to lift the fnnel, this forc should be larger than its weight mg:

$$mg = N_z = \int_0^H \rho g(H - z) \cdot 2\pi z\,dz = 2\pi\rho g \int_0^H (H - z) z\,dz = 2\pi\rho g \left(\frac{Hz^2}{2} - \frac{z^3}{3}\right)\Bigg|_0^H = 2\pi\rho g \frac{H^3}{6} \Rightarrow H = \sqrt[3]{\frac{3m}{\pi\rho}}.$$

The total volume of the liquid is determined via the expression for the special cup volume:

$$V = \frac{2}{3}\pi R^3 - V_{empty} = \frac{2}{3}\pi R^3 - \frac{1}{3}\pi(R - H)^2(2R + H) = \frac{\pi}{3}\left[H^3 - 3R^2 H\right] = \frac{\pi}{3}\left[\frac{3m}{\pi\rho} - 3R^2\sqrt[3]{\frac{3m}{\pi\rho}}\right].$$

For part two (4 marks), there are several approaches. One is to consider forces acting on a small piece of liquid near the surface at radius r. Another way is to start with the fact that the liquid surface always orients perpendicular to the gravity direction. For the rotating fluid, gravity is effectively replaced by the acceleration of the non-inertial reference system. Refer to the figure below for plotting and vectors directions. From figure, one can find

$$\tan\alpha = \frac{a_{cf}}{g} = \frac{\omega^2 r}{g}.$$

In its turn, the same angle is between a tangent to the surface and a horizontal line. So, the expression for $\tan\alpha$ can be treated as a derivative for the surface function. Therefore, the equation of the surface is given as (axial symmetry is automatically accounted):

$$z(r) = \int \frac{\omega^2 r}{g} dr = \frac{\omega^2 r^2}{2g}.$$

If locating the origin at the lowest point of the free surface, the integration constant sets to be zero.

For part three (2 marks), as the liquid is at boiling point, all heat goes for evaporation. We can express a small portion of heat in two ways: as it comes from the heater, and as it is spent on evaporation

$$dQ = Pdt = q\,dm.$$

The evaporated mass can be estimated as a liquid disk, where the thickness is small and radius is a cup radius (a in notations of formulae in the main text):

$$dm = \rho dV = \rho\pi a^2 dz = \rho\pi(R^2 - z^2)dz.$$

Joining together, we get the differential equation for the parameter $z(t)$:

$$\rho q\pi(R^2 - z^2)\,dz = P\,dt,$$

$$\frac{dz}{dt} = \frac{P}{\pi\rho q(R^2 - z^2)}.$$

This equation is solved by variables separation. Given the expression for a, at the initial moment $z(t = 0) = 0$:

$$\int_0^R R^2 - z^2\,dz = \int_0^t \frac{P}{\pi\rho q}\,dt.$$

$$(R^2 - z^2/3)z\Big|_0^R = R^3 - R^3/3 = \frac{Pt}{\pi\rho q}.$$

$$t = \frac{2\pi\rho q}{3P}R^3$$