Bending Light

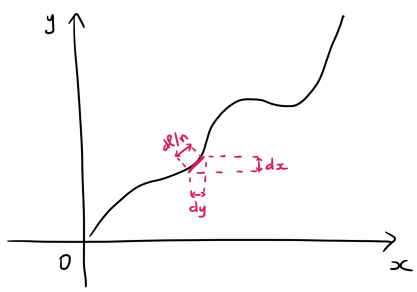
Imagine living in a (two-dimensional) world where the refractive index varies as a function of height:

$$n(y) = \sqrt{1 + e^{-y/2}}$$

A person in this world stands at the origin ($x = 0, y = 0$) and shines a light beam directed at 45° upward.
(20 marks) Determine the equation of the path of the light beam. (Hint: light always travel in the path that takes the least time.)

Solution: In a dense medium, the speed of light is lowered by a factor of $1/n$. An equivalent way to look at this is by imagining that the speed of light remains the same, but the distance is dilated. We call this distance *optical path lenth*. Suppose the light beam travels along a trajectory given by $y(x)$. We consider an infinitesimal segment of the graph (illustrated in the figure); the infinitesimal optical path length is

$$d\ell = n(y)\sqrt{dx^2 + dy^2}.$$



Thus, our task is to minimize the total optical path length:

$$\ell = \int n(y)\sqrt{dx^2 + dy^2} = \int n(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

[1 mark]

We can define the functional

$$L(x, x', y) = n(y)\sqrt{1 + x'^2},$$

where $x' = dx/dy$ and evoke the Euler-Lagrange equation. This yields

$$\frac{d}{dy}\left(\frac{n(y)x'}{\sqrt{1 + x'^2}}\right) = 0.$$

[3 marks]

Hence,

$$\frac{n(y)x'}{\sqrt{1+x'^2}} = \text{constant}.$$

[2 mark]

The constant on the RHS can be determined by noting that, at $y = 0$, $n(y = 0) = \sqrt{2}$ and $\frac{dx}{dy}|_{y=0} = 1$. Hence the constant is

$$\frac{\sqrt{2}(1)}{\sqrt{1+1}} = 1.$$

We are left to solve:

$$\frac{n(y)x'}{\sqrt{1+x'^2}} = 1.$$

With a little algebraic manipulation:

$$(n^2 - 1)x'^2 = 1,$$

or, using the given expression for $n(y)$:

$$e^{-y}\frac{dx}{dy} = 1,$$

but this is quite simple; the solution is

$$x = \int e^y dy = e^y + C$$

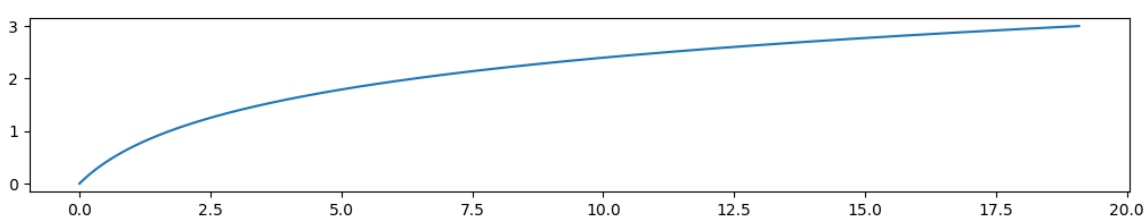
[2 marks]

Where C is a constant to be determined from the initial condition: when $y = 0$, $x = 0$. Hence, $C = -1$. The final solution is

$$\boxed{x = e^y - 1}$$

[1 mark]

Below is a quick plot of the trajectory



[1 mark]